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Abstract
The dynamics of dark solitons is studied within the framework of the mathematical model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. A comparative analysis of the solution of no stationary problems is performed for a linear harmonic oscillator and the NSE model with a harmonic potential for different signs of the self-action potential. It is shown that the main specific feature of the dynamics of dark NSE solitons in a parabolic trap is the formation of solitons with dynamically changing form factors producing the periodic variation in the modulation depth (the degree of "blackness") of dark solitons. The oscillation period of the dark solitons does not coincide with the oscillation period of a linear quantum-mechanical oscillator, which is caused by the periodic transformation of the black solitons to the gray one and vice versa.

Keywords: solitons, nonlinear Schrödinger and Gross-Pitaevsky equations, Bose-Einstein condensate.

1 Dark optical solitons in optical fibers and dark soliton waves of matter in a Bose-Einstein condensate

Numerous experiments have shown that the nonlinear dynamics of a Bose-Einstein condensate (BEC) in magnetic traps is well describer by the mathematical model of the average Gross-Pitaevsky field for the wave function of the condensate [1-4]. Analysis of the dynamic analogy between the BEC of photons, the atomic condensate and the condensate of cooper pairs in a semiconductor performed by Oraevsky [5-10] have shown that an ensemble of a sufficiently large number of the particles in the BEC behave as a classical field having the amplitude and phase. The condensate dynamics can be considered as an essentially nonlinear process, which is completely similar to the formation of the BEC of photons in a laser (see, for example, review [6] and references therein).

Indeed, as has been pointed out in pioneering experimental papers on the generation of solitons in a BEC [11-14] , there exists a deep mathematical analogy between the theory of soliton waves of matter and the theory of optical solitons in optical fibers (see, for example, papers [11-17] and monographs [18-20] and references therein). Note that nonlinear collective excitations were first discovered in a BEC in the so-called cigar-shaped traps with the transverse size much smaller than the longitudinal size [11-14]. In this case, the Gross-Pitaevsky equation is substantially simplified [1-4].

$$i\frac{\partial u}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}+R\left|u\right|^2 u-\frac{1}{2}\Omega^2\tau^2 u=0 \tag{1}$$

Depending on the sign of the no linearity parameter R (corresponding to different physical scenarios of the nonlinear interaction of atoms: attraction or repulsion), either bright or dark soliton waves of matter are formed [11-17]. The experimental discovery of the soliton waves of matter in a BEC [11-14] (dark for $R<0$ and bright for R>0) was simulated to a great degree by the fact that model (1) in the limiting case $\Omega=0$ transforms to the well-studied NSE describing, in particular, optical solitons of the envelope of an electromagnetic field [18-26] predicted by Hasegawa and Tappert [23] long the before the development of low-loss fibers, and also bright solitons discovered in [24] and dark solitons discovered in papers [25-26].

The main goal of this papers is to study the dynamics of dark solitons within the framework of the mathematical NSE model within an external harmonic potential. The features of the formation and interaction dynamics of bright solitons in a harmonic trap potential were considered in the first part of our paper [27].

Let us write first of all the exact solutions of Eqn(1) for dark solitons. For $\Omega=0$ and R=1, Eqn (1) can be written in the form of the well-known NSE for the complex conjugate function u^*, which is convenient for analyzing solitons in a BEC.

$$i\frac{\partial u^*}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u^*}{\partial \tau^2}-\left|u^*\right|^2 u^*=0 \tag{2}$$

Following pioneering work [23] we represent solution of (2) in the form

$$u(\xi,\tau) = \rho^{1/2}(\tau)\exp\{\, i\sigma(\xi,\tau)\,\} \tag{3}$$

where the real (amplitude) and complex (phase) parts are described by the expressions

$$\rho(\tau) = \rho_0\left[1 - a^2 \operatorname{sech}^2\left\{\rho_0^{1/2}\, a\,\tau\right\}\right] \tag{4}$$

$$\sigma(\xi,\tau) = \rho_0\sqrt{1-a^2}\;\tau + \operatorname{arctg}\left[\frac{a}{\sqrt{1-a^2}}\, th\left(\rho_0^{1/2} a\,\tau\right)\right] + \frac{\rho_0}{2}\left(3-a^2\right)\xi \tag{5}$$

Unlike a bright soliton

$$u(\xi,\tau) = \eta\ \operatorname{sech}[\eta(\tau - q_0 + \delta\xi)]\ \exp\left[i\left(\eta^2 - \delta^2\right)\xi/2 - i\delta\tau\right] \tag{6}$$

a dark solitons (4-5) has the additional parameter *a* which determines the modulation depth [the hole depth in the intensity (4)] and phase (5) of the soliton. An important feature of the dark soliton is the time dependence of its phase (5) the phase modulation. When *a* = 1, the dark soliton becomes black,

$$u(\xi,\tau) = \rho_0^{1/2}\tanh\left(\rho_0^{1/2}\tau\right)\exp(i\rho_0\xi) \tag{7}$$

with the phase shift by π at the central part of the pulse. Because function (7) is two -valued by definition (taking the root), $u(\xi,\tau) = \pm\eta\tanh(\eta\tau)\exp\left(i\eta^2\xi\right)$, it has two singularities of opposite signs (jumps at $\pm\eta$) at the boundary conditions $\tau \to \pm\infty$ and describes the so-called topological soliton and antisoliton with topological charges +1 an -1. Solitons with identical topological charges are repulsed and with opposite charges are attracted.

Note that the concept of a topological soliton has become now generally accepted and combines the whole family of solitons discovered in various field of physics, for example, dislocations in crystals, kinks in the field theory, fluxons in Josephson junctions, waves in ferromagnetics, 2π -pulses in quantum electronics and nonlinear optics, dark solitons in optical fibers, and dark solitons of matter in a BEC.
It is interesting to follow the history of the appearance of the concept of a topological solitons. Skyrme [28-29] was the first to attempt in 1959-1962 to construct the integrated nonlinear model of the field theory

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + m^2\sin\Phi = 0 \tag{8}$$

The "jargon" name of the model- the sine-Gordon equation was yet absent then. It appeared later in the paper of Rubinstein [30], who nevertheless pointed out in the footnote on the second page that this name of the equation was given by Martin Krustal. However, the first three references in [30] are the Skyrme papers. Skyrme performed in fact one of the first numerical experiments in the field theory and observed the elastic interaction of particles, for which the managed to find absolutely correct analytic expressions. Which are now known as kinks and breathers. The term soliton appeared only for years later, but Skyrme observed the mutual repulsion between kinks and the mutual attraction between kinks and antikinks in his numerical experiments on the Herwell Mercury Computer already in 1958-1962

$$\Phi(x,t) = 4arctg \exp\left[\pm \frac{m}{\sqrt{1-\upsilon^2}}(x-\upsilon t)\right], \tag{9}$$

and assigned to kinks and antikinks the topological charges +1 and -1, respectively, by defining them by the normalization conditions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial \Phi(x,t)}{\partial x} dx = \frac{1}{2\pi}\left[\Phi(\infty,t) - \Phi(-\infty,t)\right] = N_i \tag{10}$$

The concept of the topological charge was introduced first of all because of the nature of forces between solitons, as we say now [32], observed in the numerical experiments of Skyrme. The second important circumstance is the different asymptotic of solutions at $\pm\infty$. A kink this term was introduced by Finkelstein in 1966 [32] changes from 0 to 2π and has a discontinuity in the boundary conditions. In addition, it is difficult to introduce the concept of vacuum in model (8). Indeed, the Hamiltonian of equ. (8)

$$H = \frac{1}{2}\left(\Phi_t^2 + \Phi_x^2 + 4m^2 \sin^2 \frac{1}{2}\Phi\right) \tag{11}$$

has a minimum in the vacuum states $\Phi = 2\pi n$. Therefore, the concept of numerous vacuums appears, which are defined by integers $\Phi = 2\pi n$, and topological charges are in fact the "jumps" [(normalized by condition (10)] between different vacuum states of the field.
A similar situation also takes place for dark NSE solitons (7), where the topological charges of the soliton and antisoliton determine the jumps between the field states $\pm\eta$ for the boundary condition al $\pm\infty$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial(\tau,\xi)}{\partial \tau} d\tau = \frac{1}{2\eta}\left[u(\infty,\xi) - u(-\infty,\xi)\right] = N_i \tag{12}$$

Therefore, solution (7) is written in the form $u(\xi,\tau) = \pm N_i \eta \tanh(\eta\tau)\exp(i\eta^2\xi)$, where $N_i = \pm 1$ is the topological charge.
The theoretical idealization requiring the presence of infinite fronts of a constant intensity for a dark soliton gives rise to competing nonlinear effects in real experiments with dark optical solitons. Because of this, the so-called base-pulse method the received wide acceptance in the experimental studies of optical solitons. This method uses a long additional base pulse on which a phase jump required for the generation of the dark soliton is produced.
The typical picture of the generation dynamics of a black soliton in the case is presented in Fig 1.

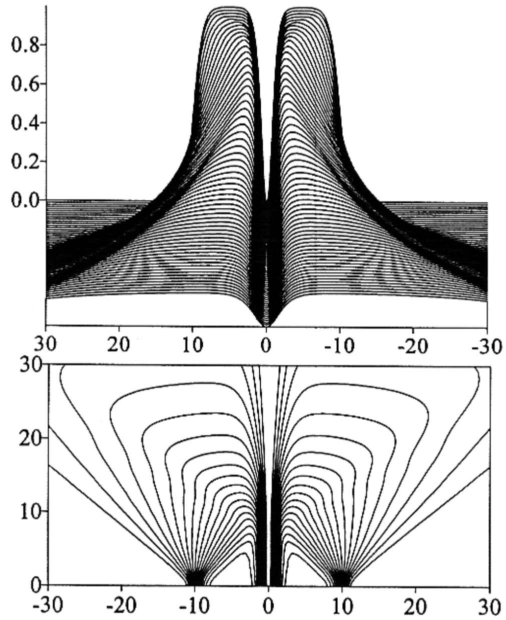

Figure 1. Dynamics of a black soliton formed on a super-Gaussian base pulse.

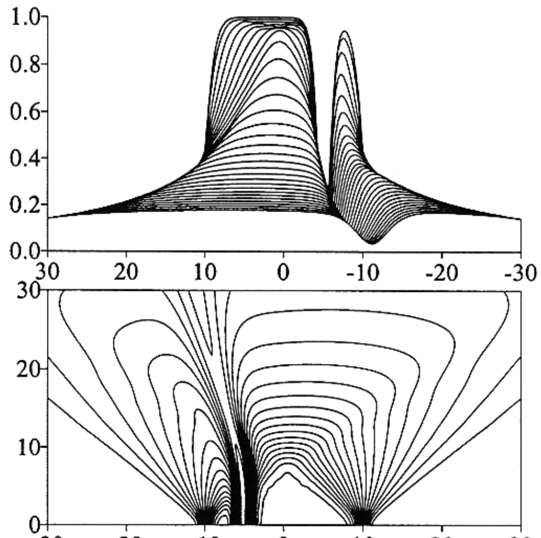

Figure 2 Effect of the base-pulse boundaries on the dynamics of a black soliton resulting in its rolling off from the base pulse and its transformation to a gray soliton.

The effect still remains not studied; however, as shown below, it plays a substantial role in analysis of the interaction dynamics of dark solitons in a BEC. Because the influence of boundaries of the base pulse cannot be accurately described by the method of inverse scattering problem [33-37], we consider the results of direct numerical simulation.

Consider the process of formation of a dark soliton on a base pulse in more detail. A finite width of the base pulse leads, first, to a substantial broadening of the dark soliton (this fact was revealed already in earlier numerical experiments [38] and, second, the combined action of dispersion and no linearity causes the flattening of the top of the base pulse, so that analytic expressions (3-6) describing the hole in the intensity against the background of an infinitely wide constant base pulse more and more correspond to the exact solution, while the fronts of the base pulse no longer affect substantially the dynamics of the dark soliton (see Figures 1 and 2)

When the center of the black pulse is shifted with respect to the center of gravity of the base the pulse, a new effect appears: the soliton pulse is accelerated and, what is the most interesting, transforms to a gray solitons because its modulation depth changes (see Fig 2). Indeed, according to expression (5), a change in the dark-soliton velocity is uniquely related to a change in its modulation depth, because according to exact topological solutions (4-5) for topological solitons, only the gray soliton moves in the coordinate system connected with the base, while the dark soliton, according to the exact solution (7), is at rest in the coordinate system connected with the group velocity of the pulse. Figure 2 demonstrates the transformation of the black soliton to the gray one in the corresponding projection.

This effect is important for the interpretation of the results of various experiments with optical solitons and solitons in a BEC. This "sliding off" of two dark solitons from the edges of the base pulse shown in Fig. 3 can be erroneously treated in a particular experiment as the repulsion interaction between dark solitons.

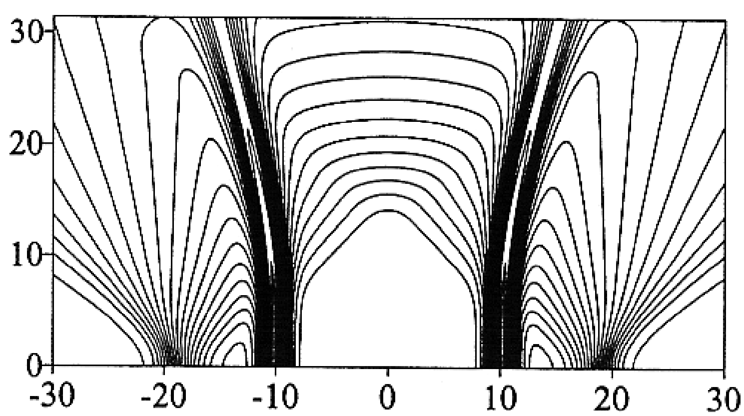

Figure 3 Two black solitons rolling off from the base pulse and transformation to grey solitons.

The general case of the formation and interaction of black and gray solitons is shown in Fig. 4.

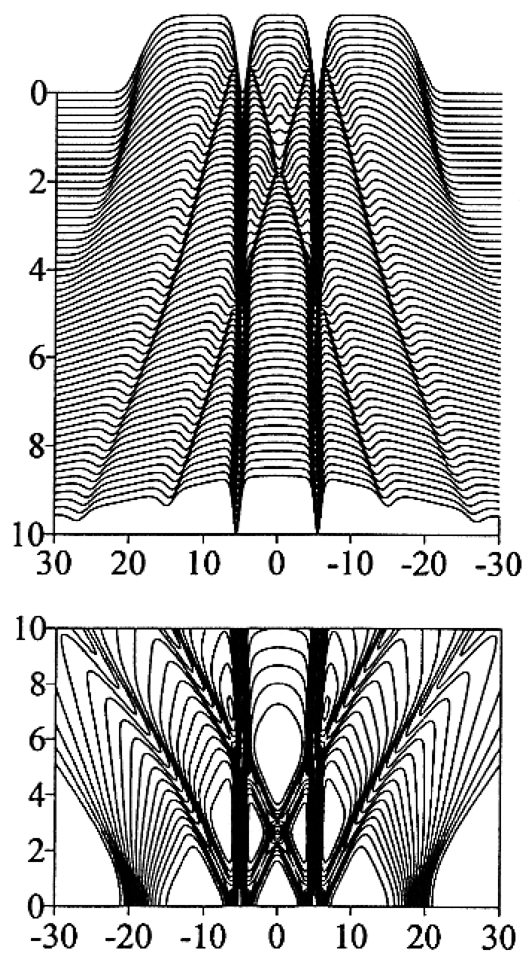

Figure 4. The interaction dynamics of black and gray solitons.

2 Basic properties of the dynamics of dark solitons in a harmonic potential

In the limit R=0, Eqn (1) is transformed to the usual Schrödinger equation for a harmonic oscillator, whose general solution for the stationary states of the oscillator is determined by the Hermitean function [39]

$$u_n(\xi,\tau)=\frac{\sqrt{\Omega}}{\sqrt{2^n n!\sqrt{\pi}}}\exp(-i\lambda\xi)H_n(\tau)\exp\left(-\frac{\Omega\tau^2}{2}\right) \qquad (13)$$

$$H_n(\tau)=\exp\left(-\tau^2\right)(-1)^n\frac{d^n}{d\xi^n}\exp\left(-\tau^2\right); \qquad \lambda=\Omega\left(n+\frac{1}{2}\right)$$

We are interested in the solution for n=1

$$H_1(\tau)=2\tau\frac{\sqrt{\Omega}}{\sqrt{2\sqrt{\pi}}}\exp\left(-\frac{\Omega\tau^2}{2}\right)\exp\left(-i\frac{3}{2}\Omega\xi\right), \tag{14}$$

which, as one can easily see, is similar to the form of the initial condition in the study of a dark (black) soliton on a Gaussian base pulse

$$u(\tau)=\eta\tanh(\eta\tau)\exp\left(-\beta\tau^2\right)\approx\tau\eta^2\exp\left(-\beta\tau^2\right) \tag{15}$$

The parameters η and β in (15) determine the duration of the dark soliton(the phase jump duration) and the width of the Gaussian base pulse, respectively. State (14), which is initially shifted from the equilibrium position, oscillates in the parabolic potential (see Fig. 5) similar to the oscillations of the bright soliton, while the interaction between two spatially separated wave function (14) is totally elastic (see Fig. 6), as in the case $n=0$.

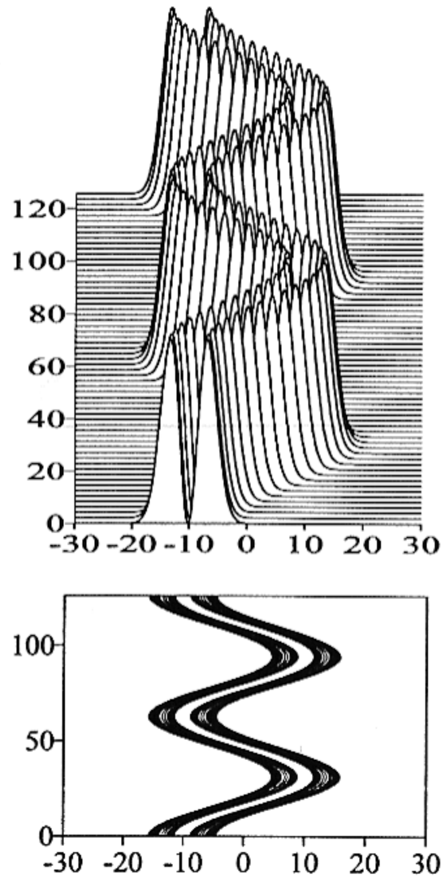

Figure 5 Spatiotemporal dynamics of the first eigenstate of an oscillator.

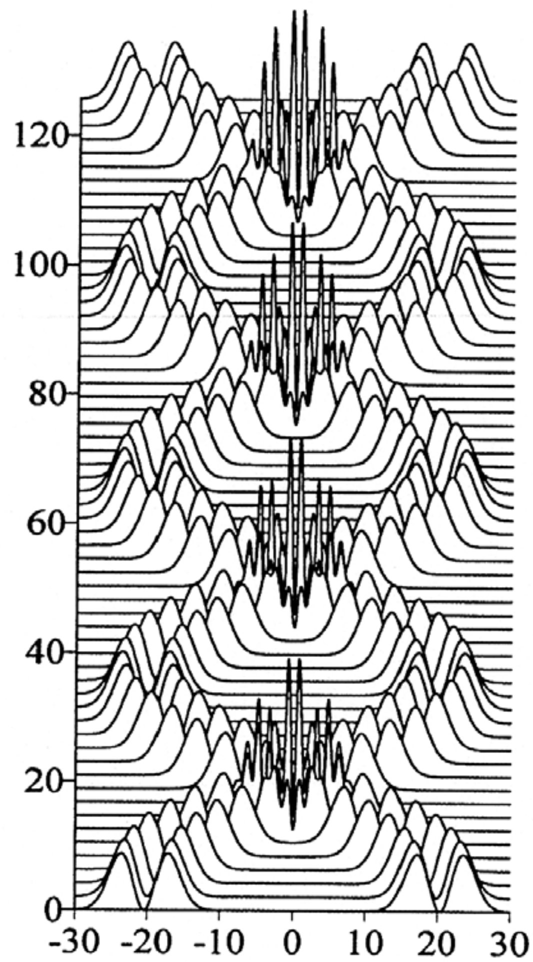

Figure 6 Interaction dynamics of two spatially separated oscillator states.

Our calculations show that the self-action effects do not prevent the formation of quasi-soliton states both for the positive and negative types of self-action
Therefore, numerical experiments performed in a broad range of variation of basic parameters of the problem reveal the following main properties of the dynamics of dark solitons in a harmonic potential:

1. The oscillation period of dark (black) solitons depends on the specific experiment. If a dark solitons is formed on a base pulse oscillating in a harmonic trap, the oscillation period virtually coincides whit the oscillation period of a linear oscillator, while the dynamics of the dark soliton is similar to that of a classical particle obeying the Newton mechanics laws (see Fig. 7).

2. In the case of an immobile base pulse with the shape periodically oscillating due to self-action effects in a harmonic potential, the oscillation period of the black soliton considerably increases because of the

periodic transformation of the black soliton to the gray one and vice versa. The soliton becomes black at the turning points in the harmonic potential and gray at the instant of its passage though the trap center, where is velocity is maximal. In the case, its oscillation period is only approximately determined by relations obtained in previous papers [15-17,20]. Typical pictures of the soliton evolution and changes in its modulation depth in this case are shown in Figs 8 and 9.

3. In the case of both positive and negative sign of no linearity, stable quasi-soliton configurations of the field can be formed, which correspond to the first stationary state of the harmonic oscillator and are characterized by virtually an elastic interaction (see Fig. 8).

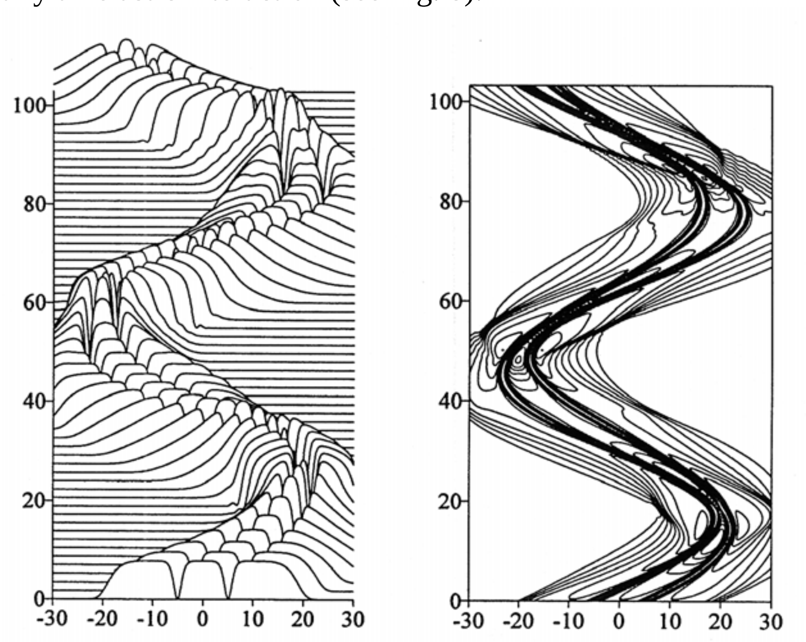

Figure 7. Dynamics of dark solitons in a parabolic trap for a base pulse oscillating in the trap

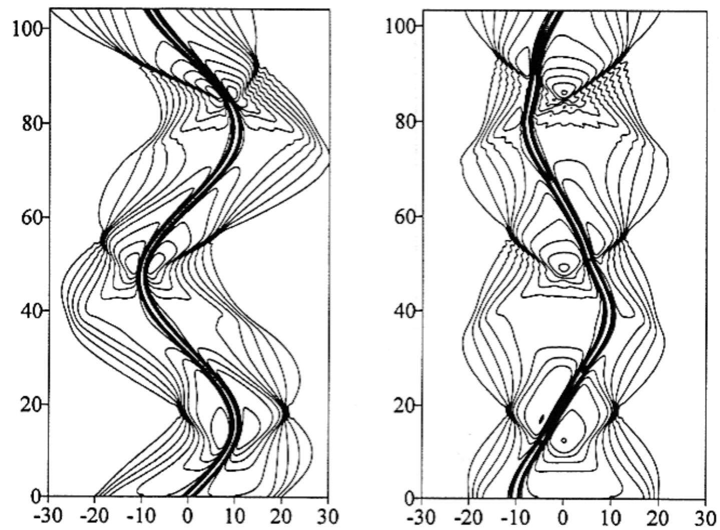

Figure 8. Change in the oscillation period of dark solitons in the parabolic potential depending on the experimental conditions (the base oscillating in the trap is compared with the base pulse at rest)

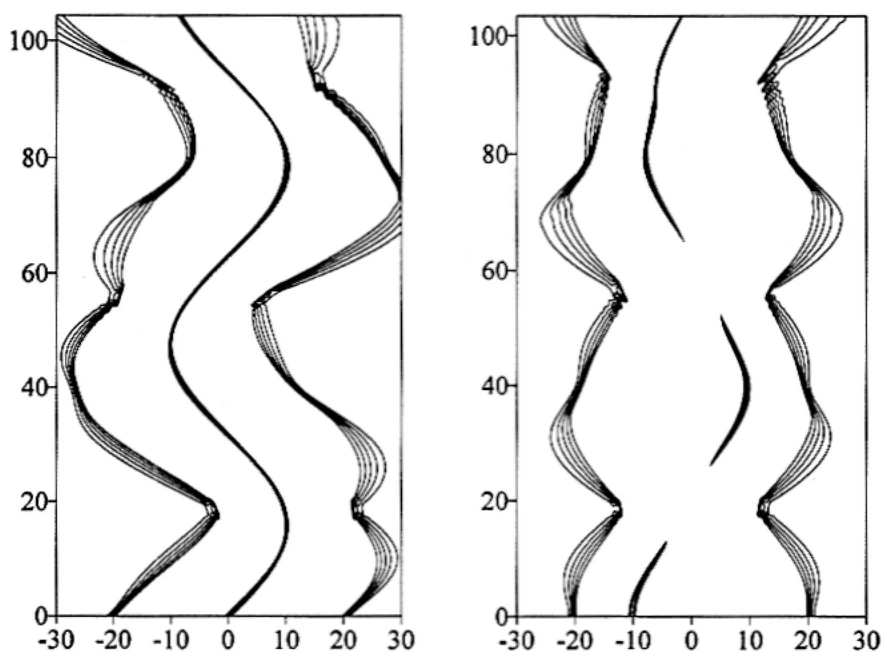

Figure 9. Effect of the periodic transformation of a black soliton to a grey one and vice versa. For the proof contour lines do not achieve the maxima of pulses but begin with the value 0.05 and come to 0.25. One can see that in the case of the oscillating base the soliton becoms black, and breaks in the contour lines at maximum velocities show that soliton became grey because its minimum intensity exeeds 0.25

Note in conclusion that our numerical experiments performed in a broad range of variation of the basic parameters of the problem allowed us to describe and explain the dynamics of solitons in the NSE model with a harmonic potential with the help of rather simple analogies and concept of quantum electronics. The use of analogies often makes it possible to solve one of the most difficult problems in the construction of a physically meaningful theory that is not restricted only to the mathematical description of one or another phenomenon. This problem is the difficulty of going from the description of a phenomenon to its explanation. An excellent example of the refined mastery of the development of physically constructive ideas, of the author's striving to propose simple and physically clear explanations based, in particular, on the well-known concepts of quantum electronics was and remain the works of Prof. A.N. Oraevsky, to whose memory this paper is devoted.

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