Internet Electronic Journal*

Nanociencia et Moletrónica

Diciembre 2005, Vol. 3, N°2, pp 573-587

BOSE-EINSTEIN CONDENSATION (BEC):

Part II. Bright solitons in BEC

C. Hernández Tenorio, J. Alcántara, L. Morales[3], R. Peña[1], Y. Reyes[1], V. Serkin[1], A. Zehe[2]

Benemérita Universidad Autónoma de Puebla, [1] Centro de Química-ICUAP
Módulo 193, Complejo de Ciencias, Ciudad Universitaria, Apdo. 1613, CP 72570
[2] Facultad de Ciencias de la Electrónica.
[3] Facultad de Ciencias Químicas.
Puebla, Pue., México; Tel. + (222) 2 29 55 00, ext. 7284
e-mail: vserkin@hotmail.com

recibido: 08 Septiembre 2005 revisado: 11. Octubre 2005 publicado: 13 Dic. 2005

Citation of the article:
C. Hernández Tenorio, J. Alcántara, L. Morales, R. Peña,Y. Reyes, V. Serkin, A. Zehe**,** Bose-Einstein-Condensation (BEC) Part II. Bright solitons in BEC, Internet Electrón. J. Nanocs. Moletrón. 2005, Vol. 3, N° 2. 573-587

copyright © BUAP 2005

BOSE-EINSTEIN CONDENSATION (BEC):

Part II. Bright solitons in BEC

C. Hernández Tenorio, J. Alcántara, L. Morales[3], R. Peña[1], Y. Reyes[1], V. Serkin[1], A. Zehe[2]

Benemérita Universidad Autónoma de Puebla, [1] Centro de Química-ICUAP
Módulo 193, Complejo de Ciencias, Ciudad Universitaria, Apdo. 1613, CP 72570
[2] Facultad de Ciencias de la Electrónica.
[3] Facultad de Ciencias Químicas.
Puebla, Pue., México; Tel. + (222) 2 29 55 00, ext. 7284
e-mail: vserkin@hotmail.com

recibido: 08 Septiembre 2005 revisado: 11. Octubre 2005 publicado: 13 Dic. 2005

Internet Electron. J. Nanoc. Moletrón. 2005, vol.3 , No 2, pags.571-587

Abstract. The dynamics of nonlinear solitary waves in studied by using the model of nonlinear Schrödinger equations(NSE) with an external harmonic potential. The model allows one to analyze on the general basis a variety of nonlinear phenomena appearing both in Bose-Einstein condensate in a magnetic trap, whose profile is describe by a quadratic function of coordinates, and an nonlinear optics, physics of lasers, and biophysics. It is shown that exact solutions for a quantum-mechanical particle in a harmonic potential and solutions obtained within the framework of the adiabatic perturbation theory of bright solitons in a parabolic trap are completely identical. This fact not only proves once more that solitons behave like particles but also that they can preserve such properties in different traps for the which the parabolic approximation is valid near potential energy minima. The conditions are found for formation of stable stationary states of antiphase soliton in a harmonic potential. The interaction dynamics of solitons in nonstationary potentials is studied and the possibility of the appearance of a soliton parametric resonance at with the amplitude of soliton oscillation in a trap exponentially increases with time is shown.

1. Introduction: Soliton history

The discovery of Bose-Einstein condensation (BEC) in trapped clouds of ultracold alkali atoms opened unique possibilities to investigate the wave nature of matter [1,2]. This has been shown in the recent BEC experiments that discover, among other things, dark and bright matter wave solitons [3-9], decay of dark solitons into vortex rings [5,10] , and soliton-vortex collisions [11-13] (see, for example, the recent review of the main experimental and theoretical achievements in this very active field of physics with coherent matter waves in [14] with an exhaustive references list).

Soliton a solitary wave with the properties of a moving elementary particle is a fundamental object of nature. Soliton appears in many different fields, including matter waves physics and applied mathematics, nonlinear quantum field theory, condensed matter and plasma physics, nonlinear spin waves in magnetic films, nonlinear optics and quantum electronics, fluid mechanics, theory of turbulence and phase transitions, biophysics, and star formation.
The history of soliton dates back to 1834, the year in which James Scott Russell observed that a heap of water in a canal propagated undistorted over several kilometers. His report, published in 1844, includes the following text:
``I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

Soliton on the Scott Russell Aqueduct on the Union Canal near Heriot-Watt University, 12 July 1995.

2. Model of the non linear Schrödinger equation with an external harmonic potential in the theory of Bose-Einstein condensation and the theory of optical solitons

The interpenetration of ideas and methods being used in various fields of science and technology becomes at present one of the decisive factors of the development of science as a whole. Among the most spectacular examples of such an interchange by ideas and theoretical methods for analysis of various physical phenomena is the problem of the dynamics of a solitary nonlinear wave described by the mathematical model of nonlinear Schrödinger equation (NSE) with an external harmonic potential. This model is used in a variety of fields of modern science and probably will be able to play the basic role similar to that played in due time by the model of a quantum-mechanical linear harmonic oscillator in the development of modern physics.

At present among the most important applications of the NSE model with a harmonic potential are the studies of nonlinear phenomena observed upon the BEC of atoms in vapours of alkaline-earth metals. It is known that the nonlinear dynamics of a BEC in magnetic traps is described by the Gross-Pitaevsky average-field model [15-16]

$$i\hbar \frac{\partial}{\partial t}\Phi(\vec{r},t) = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}) + G\left|\Phi(\vec{r},t)\right|^2\right)\Phi(\vec{r},t) \tag{1}$$

$$V_{ext}(\vec{r}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right), \quad G = \frac{4\pi\hbar^2 a}{m}$$

The nonlinear term in the equation for the wave function Φ of the condensate, where G is the energy of the pair interaction between particles, a is the interaction length, V_{ext} is the energy of the condensate interaction with the external field of the trap, the rest of the notation is standard takes into account pair interactions between the condensate particles.

The conditions of the applicability of model (1) and of the so-called average-field approximation are discussed in detail, for example, in monograph [17] and review [18]. Because the condensate contains a macroscopically large number of particles, the wave function of the condensate becomes a classical macroscopic quantity, similarly to the strength of the electromagnetic-wave field, which becomes classical for large occupation numbers of photons in each state.

Note, however, that while the problem of localization of the Bose condensate of photons has been already solved in pioneering papers of Basov and Prokhorov with co-workers (see, for example Novel lectures [19-20], pioneering papers [21-23] and references therein), the problem of the localization of a neutral atom still remains one of the complicated problems up to now. The solution this problem was first proposed by the Letokhov [24], who showed that atoms can be localized in nodes or antinodes of a standing light wave whose frequency is far from the atomic transition frequencies. At present the method of laser manipulation of an atomic condensate is generally accepted.

In the absence of an external potential, eqn (1) is the NSE, which is well studied in the theory of self-focusing. Because the one dimensional NSE belong to the class of exactly integrated equations [25] and has many exact solutions [26] the model of a condensate in the so-called cigar-shaped trap with the transverse dimensions far smaller than the longitudinal size proved to be attractive. It is in cigar-shaped traps that nonlinear collective excitation in the BEC were first discovered, which were called bright and dark soliton waves of matter; and it is in pioneering experimental studies of the generation of solitons in the BEC [3-13] that a profound mathematical analogy between the theory of solitons waves of matter and the theory of optical solitons in optical fibers (see also monographs [27-30] and comprehensive references therein).

The passage to the one-dimensional dimensionless NSE with a harmonic potential. Is described in detail. For example, in [31-32].

$$i\frac{\partial u}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}+R|u|^2u-\frac{1}{2}\Omega^2\tau^2u=0 \qquad (2)$$

Note that one-dimensional model (2) was developed in fact simultaneously in the BEC theory and the theory of optical solitons. For example, this model was considered in papers [33-34] in the development of the concept of quasi-solitons in fiber optics communication links with periodic variations in the group-velocity dispersion (this field of practical applications of solitons is discussed comprehensively in books [27,30,35]). This model also appears in the study of generation of solitons in the forbidden region of group-velocity dispersion. In [36-37], the situation was considered, in particular, when a pair of solitons was used as the trap potential. In this case, to control soliton pulses form a nearly parabolic well for a laser pulse with a different wavelength lying, for example, in a spectral region forbidden for the generation of solitons. A soliton captured in a parabolic trap not only exists in the forbidden region of parameters but also preserves its unique properties even in the femtosecond time range [38].

The above examples of using the mathematical NSE model with an external harmonic potential in the BEC theory and problems of nonlinear fiber optics by no means do not exhaust the list of possible applications of the model under study. Thus, the NSE model with a harmonic potentials opens up new possibilities in simulations of nonlinear mechanisms of energy transfer in long biological polymer molecules. The study of these mechanisms is important for the explanation of the appearance of soliton waves.

From the point of view of practical applications, one of the central problems of the theory is the search for new possibilities to control the dynamics of solitons. This determined the scope of problems that we considered in this paper. The investigation of the BEC dynamics includes the analysis of the role of boundaries of a cigar-shaped tap whose longitudinal size is assumed comparable with the region of variations in the order parameter of the BEC. The nonstationary problem of the dynamics of formation and interaction of soliton in the BEC is considered for bright, dark, and grey solitons. From the point of view of possible applications in high-speed soliton optical communication links, of practical interest can be the conditions of a complete compensation of forces between solitons discovered by us.

3. Comparative analysis of transient processes in the model of a linear harmonic oscillator and the NSE model with a harmonic potential

It is well known that the time dependences of the average values of the momentum and coordinate in the model of a linear quantum-mechanical oscillator

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + \frac{m\omega^2 x^2}{2}\Psi(x,t) \qquad (3)$$

$$\int_{-\infty}^{\infty} \left|\Psi(x,t)\right|^2 dx = 1$$

In the state with the wave function

$$\Psi_0(t=0,x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left\{-\frac{m\omega}{2\hbar}(x-x_0)^2 + i\frac{p_0 x}{\hbar}\right\} \qquad (4)$$

(where ω in the circular frequency, x is the displacement of a particle with the mass m from the equilibrium position) are determined by the well-known expressions [42]

$$\overline{x(t)} = x_0 \cos\omega t + \frac{p_0}{m\omega} sin\omega t \qquad (5)$$

$$\overline{p(t)} = p_0 \cos\omega t - m\omega x_0 \sin\omega t \qquad (6)$$

Let us show that by using the methods of the adiabatic perturbation theory of solitons [43-49], we can obtain analytic expressions for the main parameters of NSE solitons in a parabolic potential, which are completely mathematically equivalent to expression (5) and (6), and thereby approximately describe the motion of solitons as the motion of material points in the Newton mechanics.
By considering the external potential of the NSE (2) as a small perturbation

$$i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = i\varepsilon(u) \qquad (7)$$

We can write the solution of eqn (2) in the form of a soliton with dynamically changing parameters (amplitude, the centre-of mass position, phase and velocity)

$$u(\xi,\tau) = \eta(\xi)\ \mathrm{sech}[\eta(\xi)(\tau - q(\xi))]\ \exp[i\varphi(\xi) - i\delta(\xi)\tau] \qquad (8)$$

which are related by simple differential equations

$$\frac{dq}{d\xi} = -\delta\,, \qquad \frac{d\varphi}{d\xi} = \frac{1}{2}\left(\eta^2 - \delta^2\right) \tag{9}$$

Within the framework of the adiabatic perturbation theory for solitons, these for parameters are described by the systems of equations [43-49].

$$\frac{d\eta}{d\xi} = \mathrm{Re}\int_{-\infty}^{\infty} \varepsilon(u)\, u^*(\tau)\, d\tau \tag{10}$$

$$\frac{d\delta}{d\xi} = -\mathrm{Im}\int_{-\infty}^{\infty} \varepsilon(u) \tanh[\eta(\tau - q)]\, u^*(\tau)\, d\tau \tag{11}$$

$$\frac{dq}{d\xi} = -\delta + \frac{1}{\eta^2}\mathrm{Re}\int_{-\infty}^{\infty} \varepsilon(u)\,(\tau - q)\, u^*(\tau)\, d\tau \tag{12}$$

$$\begin{aligned}\frac{d\varphi}{d\xi} &= \mathrm{Im}\int_{-\infty}^{\infty} \varepsilon(u)\left\{\frac{1}{\eta} - (\tau - q)\tanh[\eta(\tau - q)]\right\} u^*(\tau)\, d\tau \\ &+ \frac{1}{2}\left(\eta^2 - \delta^2\right) + q\frac{d\delta}{d\xi}\end{aligned} \tag{13}$$

For the parabolic interaction potential, we obtain from eqns. (10-13)

$$\frac{d\delta}{d\xi} = \Omega^2 q\,, \quad \frac{dq}{d\xi} = -\delta\ , \tag{14}$$

which leads to two equations for a harmonic oscillator

$$\frac{d^2 q}{d\xi^2} = -\Omega^2 q \quad , \qquad \frac{d^2\delta}{d\xi^2} = -\Omega^2\delta\ , \tag{15}$$

whose solution have the form

$$q(\xi) = q_0\cos(\Omega\xi) - \frac{\delta_0}{\Omega}\, sin(\Omega\xi) \tag{16}$$

$$\delta(\xi) = q_0\Omega sin(\Omega\xi) - \delta_0\cos(\Omega\xi)\,, \tag{17}$$

where the parameters with the subscript ‘0’ correspond to the initial values of the velocity δ and the centre-of-mass position q.

Therefore, the main result of the approach developed in the paper in the conclusion that analytic results obtained for a nonstationary quantum-mechanical harmonic oscillator and for solitons in a harmonic trap are completely mathematical equivalent. To be certain that this is the case, it is sufficient to change the sign of the initial pulse in expression (5-6), and (16-17) and consider the results of numerical experiments.

We emphasize, however that, that while the results for linear model (3) are exact (expression (5) and (6), expression (16) and (17) for nonlinear model (7) are valid only within the framework of the adiabatic perturbation theory for solitons. Recall that the so-called adiabaticity of perturbations, which allows that use of the perturbation theory, means that a change in the soliton shape remains small during characteristic times corresponding to the length of the dispersion spread of a wave packet.

Our comparative numerical analysis of the dynamics of NSE solitons in a parabolic trap described by model (2) for $R \neq 0$ and the dynamics of a linear oscillator (equation (2) for $R = 0$) revealed a number of general qualitative properties.

Consider the typical results of numerical experiments presented in Figs 1-5 both for single and interacting wave packets. Figures 1-3 compare the dynamics of a linear oscillator, which is initially in the state with wave function (4), whose centre of gravity is initially displaced with respect to the equilibrium position, and the dynamics of the NSE soliton in a parabolic potential. The nonstationary problem for a linear oscillator, which has exact analytic solution (5-6), illustrates the possibilities and stability of the numerical algorithm (calculations were performed with a double accuracy), the contour map (equal-level lines) is represented at the logarithmic scale.

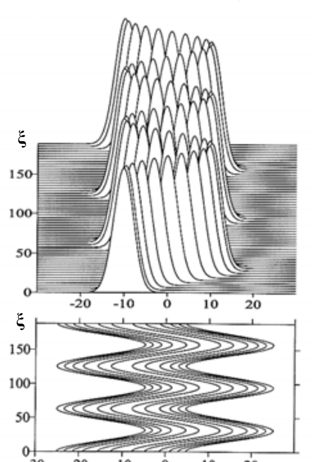

Figure 1. Spatiotemporal dynamics of the ground state of an oscillator with the centre of gravity initially displaced with respect to the equilibrium position. The contour map (equal-level lines) for the normalised wave function is presented at the logaritmic scale, beginning from the value 10^{-10} with the step 10^{2}. Calculations are performed with a double accuracy for Eqn (2) for $\Omega = 0.1$ in the absence of self-action ($R = 0$).

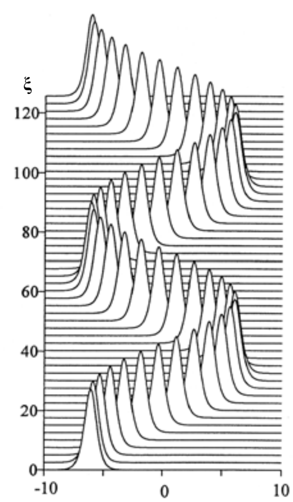

Figure 2. Nonlinear dynamics of the NSE soliton in a harmonic potencial calculated within the framework of model (2) for $\Omega = 0.1$ and $R = 1.0$.

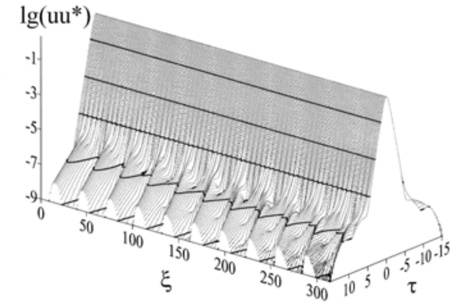

Figure 3. Soliton dynamics presented at the logarithmic scale, beginning from the value lg(uu*) = -9 with step 2.

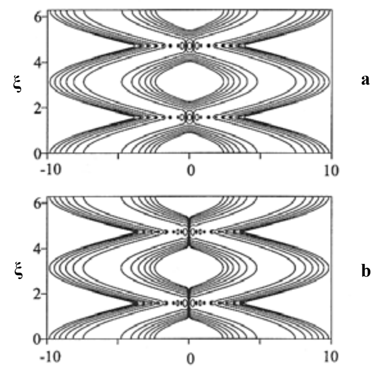

Figure 4. Contour maps (at the logarithmic scale) illustrating the dynamic of a linear harmonic oscillator initially in the states corresponding to the in-phase (a) and out-of-phase (b) wave functions. Calculations are performed for Ω = 1.0 and R = 0 for Eqn (2). The contour lines begin with tha value lg(uu*) = -6 and are drawn with the step 1.

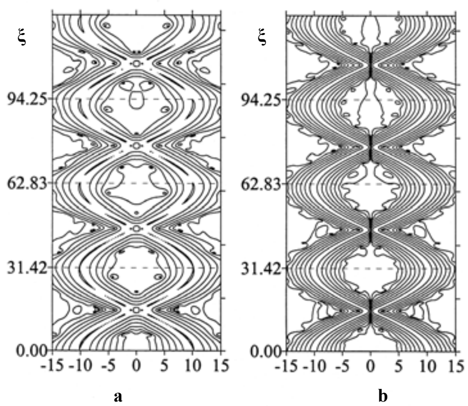

Figure 5. Contour maps (at the logarithmic scale) illustrating the interaction dynamic of in-phase (a) and out-of-phase (b) NSE solitons in a harmonic potential. Calculations are performed for Ω = 0.1 and R = 1.0 for Eqn (2). The contour lines begin with the value lg(uu*) = -6 and are drawn with step 1.

Because NSE (2) for $R = 0$ transforms to the usual Schrödinger equation for a linear harmonic oscillator, is ground-state wave function in the dimensionless form is described by the expression

$$u_0(\xi,\tau) = \left(\frac{\Omega}{\sqrt{\pi}}\right)^{1/2} \exp\left\{-\frac{\Omega}{2}\tau^2 - i\frac{\Omega}{2}\xi\right\}$$

Although the calculated dynamics of the NSE soliton in a harmonic potential presented in fig. 2 completely aggress with analytic estimates (16-17) obtained using the perturbation theory (7-13), nevertheless it does not allow us to make a certain conclusion about the real dynamics of the soliton envelope. Indeed, if we consider the distortions of the soliton shape at a ‘deeper’ logarithmic scale (Fig. 3), we will see how the distortions of the solitons shape gradually appear, which tend to accumulate, whereas the central part of the pulse does not change substantially.

The dynamics of a linear harmonic oscillator found initially in the state with the wave function representing a linear superposition

$$u_{1+2}(\xi = 0, \tau) = u_0(\tau - q_0) + u_0(\tau + q_0)\exp(i\varphi)$$

Of the two waves function separated in space, where the parameter φ describes their relative phases, is presented in Fig. 4. The contour maps of equal-level lines at the logarithmic scale clearly show that the initial state can be considered as two virtually nonoverlapping Gaussians with parameters corresponding to the wave function of the ground state of a harmonic oscillator. The qualitative picture of their interaction, which is similar due to the optical-quantum-mechanical analogy, for example, to the interaction of two Gaussian beams in a gradient waveguide, is determined by the initial phase difference. In the case of in-phase initial states ($\varphi = 0$), their interaction at the trap centre $(\xi = 0)$ corresponds to attraction, while in the case of out-of-phase states $(\varphi = \pi)$, their interaction corresponds to repulsion, which is clearly seen in the map in Fig. 4. Thus, the equal-level lines in Fig 4a intersect at the trap centre, while in Fig. 4b they do not intersect at the trap centre, by forming a gap.

It is well known that the dynamics of NSE solitons is also determined by phase relations between pulses. The attraction of i-phase solitons and repulsion of out-of-phase NSE solitons are described by analytic expressions obtained, in particular, by the methods of the adiabatic perturbation theory (all the priority papers on the interaction of solitons in the NSE model are cited, for example, in review [49]).

Let us compare the interaction dynamics of in-phase and out-of-phase NSE solitons in a harmonic potential (Fig. 5) with that of a harmonic oscillator presented in Fig. 4. The initial state is also taken in the form of two almost nonoverlapping functions, which are now represented not by two Gaussians but by two hyperbolic secants, each of them being the exact solution of the NSE in the absence of a harmonic potential. One can see at the logarithmic scale how the distortions of the soliton shape appear during the interaction of solitons, out-of-phase-phase solitons never being overlapped (by repelling). It follows from our numerical experiments performed in a broad range of variations of the parameters of the problem that the interaction of out-of-phase solitons drastically differs the dynamics of in-phase solitons, being more stable (Fig. 5).

Note by analogy that the multisoliton solutions of the Gross-Pitaevsky equation (1) are not the multiboson wave functions. For a harmonic oscillator (3), Ψ in the one-particle wave function, while the function Φ in the Gross-Pitaevsky equation is a collective variable – the order parameter reflecting the evolution of the spatial density of the condensate, its spatial argument rather than the coordinates of bosons in the condensate.

The mathematical formulation of the problem proves to be completely similar to (7-13); however, now the soliton pair

$$u_{1;2}(\xi, \tau) = \eta_{1.2}(\xi)\ \mathrm{sech}\left[\eta_{1;2}(\xi)(\tau - q_{1;2}(\xi))\right]\ \exp\left[i\varphi_{1;2}(\xi) - i\delta_{1;2}(\xi)\tau\right] \qquad (18)$$

is used as the initial condition. By substituting (18) into (7), we obtain the perturbed NSE model for two solitons

$$i\frac{\partial u_1}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u_1}{\partial \tau^2}+|u_1|^2 u_1=\frac{1}{2}\Omega^2 u_1-2|u_1|^2 u_2-u_1^2 u_2^* \tag{19}$$

$$i\frac{\partial u_2}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u_2}{\partial \tau^2}+|u_2|^2 u_2=\frac{1}{2}\Omega^2 u_2-2|u_2|^2 u_1-u_2^2 u_1^* \tag{20}$$

the three terms in the right sides of (19-20) describe the confinement (capture) of solitons by a parabolic trap and nonlinear interaction between overlapping solitons. Note that depending on the sign of the parameter Ω^2, eqns (19-20) describe either the confining (well) or repulsive (barrier) potentials. Let us also emphasize that in the absence of an external potential, the problem of the interaction of NSE solitons within the framework of the adiabatic perturbation theory has been already solved in classical papers [43-47] , while the model considered here is in fact the generalization of the previous results to the case of an external harmonic potential. After transformations similar to (7-17), we obtain finally the equation describing the interaction dynamics for a pair of NSE solitons in a harmonic potential

$$\frac{d^2 q}{d\xi^2}=-\Omega^2 q-4\eta^3\exp(-2\eta q)\cos(\varphi) \tag{21}$$

where $q=(q_2-q_1)/2$ is the distance and $\varphi=(\varphi_1-\varphi_2)/2$ in the phase between solitons.

The most interesting results following from the analysis of our analytic model are:

1.- The oscillation period of a pair of solitons in a parabolic trap is determined by the combined action of two force4s. The first force increases linearly with distance and dominates at large distances between solitons. The second force is a nonlinear short-range force (exponentially decreasing with distance) and depends on the phases of interacting solitons. It begins to play the role only when the wave functions are well overlapped and solitons closely approach each other.

2.-The phase dependence of forces and the sign of the potential (attraction or repulsive external potential) allow the efficient control of the dynamics of Schrödinger solitons. When these two forces are exactly compensated, for example, for out-of-phase solitons in the attraction potential or for in-phase solitons in the repulsive potential, a stationary state can be formed. The study of the stability of the stationary regime by the usual method of linearization of equations with respect to stationary values proves to be quite simple and shows that a stable state forms only out-of-phase solitons in the attraction potential , while the bound states of solitons in the repulsive potential are always unstable.

These forces substantially change the interaction dynamics of in-phase solitons in a parabolic trap under the condition

$$\Omega^2\le\frac{4}{q}\eta^3\exp(-2\eta\, q) \tag{22}$$

because in the canonical case (without potential), the oscillation period of the soliton pair with parameters $q = q_0$ and $\eta = 1$ is determined by the relation

$$T_{sol} = \frac{\pi}{2} \exp(q_0) \tag{23}$$

Inequality (22) relates the main parameters of the systems

$$\frac{T_{sol}}{T_0} \geq 2\sqrt{q_0} \tag{24}$$

where T_0 is the oscillation period of a harmonic oscillator.
The interaction forces between two solitons are exactly compensated if

$$\Omega_0^2 = -\frac{4}{q}\eta^3 \exp(-2\eta\, q) \cos(\varphi) \tag{25}$$

This gives, in particular, the condition for formation of a stable stationary state for out-of-phase solitons with parameters $q = q_0$ and $\eta = 1$,

$$\frac{T_{sol}}{T_0} = 2\sqrt{q_0} \ , \tag{26}$$

and the critical frequency of the harmonic potential

$$\Omega_0^2 = \frac{4}{q}\eta^3 \exp(-2\eta\, q) \tag{27}$$

4. Nonstationary potential. Parametric resonance for solitons in a harmonic potential

The assumed adiabaticity of the NSE soliton dynamics in a parabolic trap allows one to consider more complicated processes appearing in nonstationary harmonic potentials, when the parameter Ω depends on time. By using (7-17), we can easily obtain the equation for the coordinate of the centre of gravity of a soliton:

$$\frac{d^2 q}{d\xi^2} + \Omega^2(\xi)\, q = 0 \tag{28}$$

It is well known that eqn (28) appears in the theory of unclosed oscillatory systems in witch the external action is reduced to temporal variations in the parameters of the systems [50]. A simple example of such a system is a mathematical pendulum with the point of support experiencing a specified periodic motion in the vertical direction. When the function Ω is periodic, the so-called parametric resonance can appear in the system described by eeqn (28). This means that the state of rest oscillatory system in the equilibrium position becomes unstable – an arbitrary small deviation from this state rapidly increases with time. The condition of the appearance of the parametric resonance, when the function

$$\Omega^2(\xi) = \omega_0^2(1 + h\cos\gamma\xi) \tag{29}$$

weakly differs from the constant ω_0^2, where studied in detail, for example, in [50]. It was shown that the parametric resonance is most intense when the perturbation frequency in close to the double frequency: $\gamma = 2\omega_0 + \varepsilon$.

The solution of the Mathieu equation of motion

$$\frac{\partial^2 q}{\partial\xi^2} + \omega_0^2\left[1 + h\cos(2\omega_0 + \varepsilon)\xi\right]q = 0 \tag{30}$$

gives the conditions for the appearance of the parametric resonance in the frequency interval $-\frac{1}{2}h\omega_0 < \varepsilon < \frac{1}{2}h\omega$ with the parameter $s^2 = \frac{1}{4}\left[(h\omega_0/2)^2 - \varepsilon^2\right]$ of exponential amplification of oscillations (we follow here paper [50]). It is known that the parametric resonance also takes place at the frequencies $2\omega_0/n$, where n is an integer. However, the widths of resonance instability region rapidly decreased with increasing n. The amplification parameter s also decreases [50].

Therefore, eqn (28) and the mathematical analogy with the parametric resonance suggest the possibility of excitation of parametric resonance also in the NSE model with a nonstationarity harmonic potential (29). This conclusion has been performed by direct numerical calculations.

5.Conclusions

In this paper, we have studied new possibilities for controlling soliton parameters produced in nonstationarity potentials. In particular, we predicted the possibility of a soliton parametric resonance, when the amplitude of soliton oscillations in a trap increases exponentially with time. The investigations of the dynamics of dark solitons will be summarized in the third part of the paper. By using the mathematical apparatus developed for applications in high-speed fiber-optics communication links [39-40], we discovered a new class of mathematical NSE models, which are exactly integrated by the method of this inverse scattering problem. The corresponding solutions will be also presented in the third part of this paper. Note in conclusion that in practically interesting cases, as a rule, different potentials can be expanded near the minima of the potential energy in a series corresponding to the harmonic approximation, so that the dynamics of solitons near the minimum of the potential energy will obey the laws considered above.

References

[1]. E.A. Cornell and C.E. Wieman, Rev. Mod. Phys. **74**, 875 (2002).
[2]. W. Ketterle, Rev. Mod. Phys. **74**, 1131 (2002).
[3]. S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G.V. Shlyapnikov, and M. Lewenstein. Phys. Rev. Lett. **83**, 5198 (1999).
[4]. J. Denschlag, J. E. Simsarian, D. L. Feder, C.W. Clark, L.A. Collins, J. Cubizolles, L. Deng, E.W. Hagley, K. Helmerson, W.P. Reinhardt, S.L. Rolston, B.I. Scneider, and W.D. Phillips. Science **287** , 97 (2000).
[5]. B. P. Anderson, P. C. Haljan, C. A. Regal, D.L. Feder, L.A. Collins, C.W. Clark, and E.A. Cornell. Phys. Rev. Lett. **86** , 2926 (2001).
[6]. K. Bongs, S. Burger, D. Hellweg, M. Kottke, S. Dettmer, T. Rinkleff, L. Cacciapuoti, J. Arlt, K. Sengstock, and W. Ertmer, J. Opt. B: Quantum Semiclass. Opt. **5**, 124 (2003).
[7]. K. E. Strecker, G.B. Partridge, A.G. Truscott, and R.G. Hulet, Nature (London), **417**, 150 (2002).
[8]. L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L.D. Carr, Y. Castin, and C. Salomon, Science **296**, 1290 (2002).
[9]. K. E. Strecker, G.B. Partridge, A.G. Truscott, and R.G. Hulet, New J. Phys. **5**, 73 (2003).
[10]. Z. Dutton, M. Budde, C. Slowe, and L.V. Hau, Science **293**, 663 (2001).
[11]. N.S. Ginsberg, J. Brand, and L.V. Hau, Phys. Rev. Lett. **94**, 040403 (2005).
[12]. T.P. Simula, P. Engels, I. Schweikhard, E.A. Cornell, and R.J. Ballagh, Phys. Rev. Lett. **94**, 080404 (2005).
[13]. S. Komineas and J. Brand, Phys. Rev. Lett. **95**, 110401 (2005).
[14]. K. Bongs and K. Sengstock, Rep. Prog. Phys. **67**, 907 (2004).
[15]. Gross E.P., Nuevo Cimento, **20**, 454 (1961).
[16]. Pitaevskii L. P., Zh. Eksp. Teor. Fiz., 40, 646 (1961).
[17]. Lifshits E. M., Pitaevsky L. P., *Statistical physics,* London. Pergamon Press, 1980; Moscow, Nauka, 1978) part 2.
[18]. Pitaevsky L. P., Usp. Fiz. Nauk **168**, 641 (1998)
[19]. Basov N.G., Semiconductor lasers, Nobel Lecture, December 11, 1964 in Nobel Lectures Physics, 1963-1970, page 89, Elsevier Publishing Company, Amsterdam.
[20]. Prokhorov A.M., Quantum electronics, Nobel Lecture, December 11, 1964 in Nobel Lectures Physics, 1963-1970, page 110, Elsevier Publishing Company, Amsterdam.
[21]. Shawlow A.L., Townes C.H,. Phys. Rev., **112**, 1940 (1958).
[22]. Basov . N. G., Prokhorov A. M., Zh. Eksp. Teor. Fiz , **101,** 431 (1954).
[23]. Basov N. G., Krokhin O. N., Popov Yu. M. Usp. Fiz. Nauk, **72**, 161 (1960).
[24]. Letokhov V. S., Pis´ma Zh. Eksp. Teor. Fiz, **7**, 348 (1968).
[25]. Zakharov V.E., Shabat A.B., Zh. Eksp. Teor. Fiz., 61, 118 (1971).
[26]. Zakharov V. E., Manakov S. V., Novikov S. P., Pitaevskii L. P., *Teoriya Solitonov, Metod Obratnoi Zadachi* (*Theory of Solitons, The Inverse Problem Method*) (Moscow: Nauka, 1980).
[27]. Hasegawa A., Kodama Y., *Solitons in Optical Communications* (Oxford University Press, Oxford, 1995).
[28]. Hasegawa A., Matsumoto M., *Optical Solitons in Fibers* (Springer, 2003).
[29]. Agrawal G. P., *Nonlinear Fiber Optics* (Academic Press, 2001).
[30]. Kivshar Yu.S., Agrawal G.P., *Optical Solitons-From Fibers to Photonic Crystals, Academic Press, 2003.*

[31]. Perez-Garcia V.M., Michinel H., Herrero H., Phys. Rev. **A 57**, 3837 (1998).
[32]. V.A.Brazhnyi, V.V.Konotop., Phys. Rev. **A68**, 043613, 2003.
[33]. Kumar S., Hasegawa A., Opt. Lett., **22**, 372 (1997).
[34]. Hasegawa A., Physica D, **123**, 267 (1998).
[35]. Hasegawa A., Ed. Massive WDM and TDM *Soliton Transmission Systems (Kluwer Academic Publishers, 2000).*
[36]. Hook A., Anderson D., Lisak M., Serkin V.N., Afanasjev V.V., Electron. Letters, **28,** 1797 (1992).
[37]. Hook A., Anderson D., Lisak M., Serkin V.N., Afanasjev V.V., J. Opt. Soc. Am B, **10**, 2313 (1993).
[38]. Hook A., Serkin V.N., IEEE J. of Quantum Electronics, **30**, 148 (1994).
[39]. Serkin V. N., Hasegawa A., Phys. Rev. Lett. , **85**, 4502 (2000).
[40]. Serkin V. N., Hasegawa A., Pis'ma, Zh. Eksp. Teor. Fiz., **72**, 125 (2000).
[41]. Serkin V. N., Hasegawa A., IEEE Journal of Selected Topics in Quantum Electronics, **8**, 418 (2002).
[42]. Landau L. D., Lifshits E. M., *Quantum Mechanics*. (Oxford: Pergamon Press. 1977; Moscow: Fismatgiz, 1963).
[43]. Karpman V. I., Pis´ma Zh. Eksp. Teor. Fiz, **25**, 296 (1977).
[44]. Karpman V.I., Maslov E.M., Pis´ma Zh. Eksp. Teor. Fiz, **73**, 537 (1977).
[45]. Karpman V.I., Maslov E.M., Phys. Lett. **A61**, 355 (1977).
[46]. Karpman V.I., Physica Scripta **20**, 462 (1979).
[47]. Karpman V.I., Solov'ev V.V., Physica **3D**, 487 (1981).
[48]. Gordon J.P., Opt.Lett., **8**, 596 (1983).
[49]. Desem C., Chu P.L., In *Optical solitons - theory and experiment, by J. R. Taylor,* (Ldn (Cambridge Univ. Press, 1992). стр. 107.
[50]. Landau L. D., Lifshits E. M., *Theoretical Physics, Vol. 1. Mechanics,* Moscow: Nauka, 1973.