Internet Electronic Journal*

Nanociencia et Moletrónica

Julio 2006, Vol.4, N°2, pp 661-668

Matter Wave Solitons Management: the Interplay between Variations of the Scattering Length and Confining Potential

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recibido: 23 de Noviembre 2005 revisado: 11 de Febrero 2006 publicado: 30 de Julio 2006

Citation of the article:
V.N. Serkin[a]*, Akira Hasegawa[b] and T.L.Belyaeva, Matter Wave Solitons Management: the Interplay between Variations of the Scattering Length and Confining Potential, Internet Electron. J. Nanocs. Moletrón. 2006, vol. 4 , No 2, pags.661-668

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Novel stable bright and dark "soliton islands" in a "sea of nonlinear matter waves" are predicted. It is shown that solitons exist only under certain conditions and both the time-dependent scattering length and the time-dependent confining potential cannot be chosen independently; they satisfy the exact integrability conditions. Matter wave solitons management concept is justified and novel opportunities for controllable creation of bright and dark solitons and their optimal compression in Bose-Einstein condensates are considered. Novel soliton solutions substantially extend the concept of solitons and generalized it to the exactly integrable models where the accelerated motion of a soliton in an external potential and its reflection from the potential boundaries are permitted.

PACS numbers: 03.75.Lm, 03.65.Ge, 32.80.Pj, 05.45.Yv

Novel stable bright and dark "soliton islands" in a "sea of nonlinear matter waves" are predicted. It is shown that solitons exist only under certain conditions and both the time-dependent scattering length and the time-dependent confining potential cannot be chosen independently; they satisfy the exact integrability conditions. Matter wave solitons management concept is justified and novel opportunities for controllable creation of bright and dark solitons and their optimal compression in Bose-Einstein condensates are considered. Novel soliton solutions substantially extend the concept of solitons and generalized it to the exactly integrable models where the accelerated motion of a soliton in an external potential and its reflection from the potential boundaries are permitted.

The discovery of Bose-Einstein condensation (BEC) in trapped clouds of ultracold alkali atoms opened unique possibilities to investigate the wave nature of matter [1, 2]. This has been shown in the recent BEC experiments that discover, among other things, dark and bright matter wave solitons [3 − 9], decay of dark solitons into vortex rings [5, 10], and soliton-vortex collisions [11 − 13] (see, for example, the recent review of the main experimental and theoretical achievements in this very active field of physics with coherent matter waves in [14] with an exhaustive references list).

Soliton - a solitary wave with the properties of a moving elementary particle - is a fundamental object of nature. Being a product of the high-speed computer revolution of the 20th century [15], the soliton paradigm is "*e pluribus unum*" and will play the major role in the unity in the science of physics [16]. Mathematical similarity between the Gross-Pitaevskii equation (GPE, [17]) governing the nonlinear excitations in BEC and the famous nonlinear Schrödinger equation (NLSE, [18]) came to be regarded as the cornerstone of BEC physics and has brought together physicists from different areas, in particular, from atomic, optical and condensed matter physics, fluid mechanics and fundamental general and particle physics.

In the low temperature regime the GPE model is the main approximation for the condensate wave function $\Psi(r,t)$ [17]:

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t)=\left(-\frac{\hbar^2\nabla^2}{2m}+\frac{4\pi\hbar^2 a_s}{m}\left|\Psi(r,t)\right|^2+V_{ext}(r)\right)\Psi(r,t) \tag{1}$$

where standard notations are used, and $r=(x,y,z)$ is the displacement from the trap $V_{ext}(r)$ center, and m refers to the mass of the atom. The nonlinearity in BEC arises from the density dependent mean field interactions between atoms proportional to the s-wave scattering length a_s [17]. The magnetic field dependence of the scattering length a_s has a resonance structure near the so-called Feshbach resonances [19], that provides new opportunities for the study of condensate physics with a very strong, very weak, positive, negative, or even time-dependent interaction strength, all within a single experiment [19].

Nonlinear dynamics of quasi-one-dimensional (cigar-shaped [20]) condensate under much greater transverse than axial confinement can be considered in the framework of the separable wave function $\Psi(r,t)=\Xi(r_\perp)$ $Q(x,t)$ [20]. In this case the GPE model (taking into account adiabatic variations both the s-wave scattering length $a_s(t)$ and confining harmonic potential $V_{ext}(x,t)$) can be reduced to the generalized GPE model with varying in time nonlinearity and confining potential [20]:

$$i\frac{\partial Q}{\partial t}+\frac{1}{2}\frac{\partial^2 Q}{\partial x^2}+R(t)\left|Q\right|^2Q-\frac{1}{2}\Omega^2(t)x^2Q=0 \tag{2}$$

Eq. (2) is written here in standard soliton units, as they are commonly known [14, 18].

In this Letter, we consider the generalized GPE model (2) from the integrable point of view and predict a new type of solitons adapted to confining potentials. It is shown that bright and dark soliton solutions exist only under certain conditions and that two time dependent parameter functions describing the nonlinearity $R(t)$ and confining harmonic potential $V_{ext}(x,t) = 1/2\Omega^2(t)x^2$ cannot be chosen independently; they satisfy the exact integrability conditions. *Novel soliton solutions for the model (2) substantially extend the concept of solitons and generalized it to the accelerated motion of a soliton in an external potential and its reflection from the potential boundaries.*

How can we determine whether a given nonlinear evolution equation (2) is integrable or not? The ingeneous method to answer this question - the Inverse Scattering Transform (IST) method - was discovered by Gardner, Green, Kruskal and Miura [21] and was elegantly formulated and developed in outstanding papers by Lax [22], Zakharov, Shabad [23] and Ablowitz, Kaup, Newell and Segur (AKNS) [24]. More recently, this method has been extended to the non-isospectral generalization of the Zakharov-Shabat eigen-value problem for the generalized NLSE model with varying dispersion, nonlinearity, and gain or absorption, however without considering additional external potentials [25].

As a first step in the construction of the exactly integrable model with varying in time external potential we introduce a convenient Lax pair concept. Let us represent the desired nonlinear evolution equation as the condition for integrability of a pair of linear differential equations, to which the IST method can be applied:

$$\widehat{\mathcal{F}}_t - \widehat{\mathcal{G}}_x + \left[\widehat{\mathcal{F}}, \widehat{\mathcal{G}}\right] = 0 \tag{3}$$

This equation must be valid for all values of spectral parameter Λ and is known as the generalization of Lax pair defining the set of the Dirac-type eigen-value linear matrix differential equations for scattering potential $Q(x,t)$

$$\psi_x = \widehat{\mathcal{F}}\psi(x,t), \qquad \psi_t = \widehat{\mathcal{G}}\psi(x,t) \tag{4}$$

Considering the general case of the time-dependent spectral parameter and taking matrices $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ in the form

$$\begin{aligned} \widehat{\mathcal{F}} &= -i\Lambda(T)\widehat{\sigma}_3 + \widehat{U}\widehat{\phi}, \\ \widehat{\mathcal{G}} &= \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix}. \end{aligned} \tag{5}$$

where $\widehat{\sigma}_3$ is the Pauli spin matrix, and matrices $\widehat{U}$ and $\widehat{\phi}$ are given by

$$\begin{aligned} \widehat{U} &= \sqrt{\frac{R_0}{1-\Phi(t)}} \begin{pmatrix} 0 & Q(x,t) \\ -Q^*(x,t) & 0 \end{pmatrix}, \\ \widehat{\phi} &= \begin{pmatrix} \exp[-i\Theta(x,t)/2] & 0 \\ 0 & \exp[i\Theta(x,t)/2] \end{pmatrix}. \end{aligned} \tag{6}$$

let us represent the desired AKNS elements of $\widehat{\mathcal{G}}$ matrix $\widehat{\mathcal{G}} = \sum_{k=0}^{n} G_k\Lambda^k$ with $\Lambda_t = \lambda_0(t) + \lambda_1(t)\Lambda$, where $\lambda_1(t) = \Phi_t/(1-\Phi)$, and $\Theta(x,t) = \lambda_1(t)x^2$, by

$$\begin{aligned} A &= \frac{i}{2}\frac{R_0}{1-\Phi}|Q|^2 - i\lambda_0(t)x - i\Lambda x\frac{1}{(1-\Phi)}\frac{\partial\Phi}{\partial t} - i\Lambda^2, \\ B &= \sqrt{\frac{R_0}{1-\Phi}}\left[\frac{1}{2}xQ\frac{1}{(1-\Phi)}\frac{\partial\Phi}{\partial t} + \frac{i}{2}Q_x + \Lambda Q\right]\exp\left[\frac{i}{2}\Theta(x,t)\right] \\ C &= -B^* \end{aligned} \tag{7}$$

The non-isospectral generalization for the IST method (given by Eqs. (3-7)) leads to the following exactly integrable NLSE model with parabolic time-dependent confining potential

$$i\frac{\partial Q}{\partial t}+\frac{\sigma}{2}\frac{\partial^2 Q}{\partial x^2}+\frac{R_0}{1-\Phi(t)}\left|Q\right|^2 Q-\frac{\sigma}{2}\left[\frac{1}{1-\Phi(t)}\frac{\partial^2\Phi(t)}{\partial t^2}\right]x^2 Q=0 \tag{8}$$

where the real function $\Phi(t)$ (we call it control function here) is required only to be a twice-differentiable, and parameter $\sigma = \pm 1$ is introduced to separate bright and dark soliton solutions. Fundamental bright ($\sigma = +1$) soliton solutions for Eq. (8) are represented by

$$Q(x,t)=\frac{2f_0}{\sqrt{R_0(1-\Phi(t))}}\mathrm{sech}\left[\xi(x,t)\right]\exp\left[-\frac{i}{2(1-\Phi(t))}\frac{\partial\Phi}{\partial t}x^2-i\chi(x,t)\right] \tag{9}$$

where

$$\xi(x,t)=\frac{2f_0 x}{1-\Phi(t)}+4v_0 f_0\int_{t_0}^{t}\frac{d\tau}{(1-\Phi(\tau))^2} \tag{10}$$

$$\chi(x,t)=\frac{2v_0 x}{1-\Phi(t)}+2\left(v_0^2-f_0^2\right)\int_{t_0}^{t}\frac{d\tau}{(1-\Phi(\tau))^2} \tag{11}$$

Fundamental dark ($\sigma = -1$) soliton solutions for Eq. (8) are given by

$$Q(x,t)=f_0\sqrt{\frac{\rho_0\ (1-\eta^2\mathrm{sech}^2\xi)}{R_0(1-\Phi(t))}}\ \exp\left[\frac{i}{2(1-\Phi(t))}\frac{\partial\Phi}{\partial t}\ x^2+i\chi(x,t)\right] \tag{12}$$

where

$$\xi(x,t)=\frac{f_0\ x\ \eta\sqrt{\rho_0}}{1-\Phi(t)} \tag{13}$$

$$\chi(x,t)=f_0\ x\ \sqrt{\frac{\rho_0\ (1-\eta^2)}{R_0(1-\Phi(t))}}+\arctan\left\{\frac{\eta}{\sqrt{(1-\eta^2)}}\tanh\left[\frac{f_0\ x\ \eta\ \sqrt{\rho_0}}{1-\Phi(t)}\right]\right\} \tag{14}$$
$$+\frac{1}{2}\rho_0(3-\eta^2)\ f_0^2\int_{t_0}^{t}\frac{d\tau}{(1-\Phi(\tau))^2}$$

and parameter ρ_0 designates the asymptotic value of wave intensity. Unlike a bright soliton (9-11), a dark soliton (12-14) has an additional parameter η, which determines the depth of modulation and its velocity against the background (black or gray solitons). Obviously, these novel soliton solutions reduce to that obtained earlier in the limit $\Phi(t) \equiv 0$ for canonical NLSE without confining potential [18]. Solitons exist only under certain conditions and both the time-dependence of magnetically tuned self-interaction energy of the condensate and variations of the confining potential cannot be chosen independently; they satisfy the exact integrability conditions:

$$R(t)=\frac{R_0}{1-\Phi(t)};\ \Omega^2(t)=\frac{\sigma}{1-\Phi(t)}\frac{\partial^2\Phi(t)}{\partial t^2} \tag{15}$$

We can conveniently classify different special cases of practical interest.

Case 1. Matter-wave solitons management concept. Let us rewrite the Lax pair and Eq. (8) in more general form (which is coincident with the convenient Eq. (2)) by using reduction $R(t) = (1-\Phi(t))^{-1}$

$$i\frac{\partial Q}{\partial t}+\frac{\sigma}{2}\frac{\partial^2 Q}{\partial x^2}+R_0 R(t)\left|Q\right|^2 Q+\frac{\sigma}{2}R(t)\frac{\partial^2}{\partial t^2}\left(R^{-1}(t)\right)x^2 Q=0 \tag{16}$$

where the nonlinear control function $R(t)$ is required only to be a twice-differentiable, but otherwise arbitrary function; there are no restrictions. There are then an infinite number of exact soliton solutions for Eq.

(16) of the form of bright and dark managed solitons represented by formulas (9-14) with substitution $\Phi(t) = 1 - R^{-1}(t)$. As follows from Eqs. (2) and (16) bright ($\sigma = 1$) and dark ($\sigma = -1$) fundamental soliton solutions for Eq. (2) exist if and only if

$$\Omega^2(t) = -\sigma R(t) \frac{\partial^2}{\partial t^2} \left(R^{-1}(t) \right) \tag{17}$$

Matter-wave solitons management concept must be consistent with variations of confining potential (15, 17).

Case 2. Exactly integrable NLSE model in the vicinity of the Feshbach resonance with continuously tuned scattering length. As follows from experiments [19] the resonance in the scattering length has the dispersive form

$$\frac{a_s(t)}{a_0} = 1 + \frac{\Delta_0/B_0}{1 - B(t)/B_0} \tag{18}$$

where B_0 is the Feshbach resonant magnetic field, a_0 is the value of scattering length far from resonance, and parameter Δ_0 represents the resonance width [19]. Notice, that equations (8) and (18) have one feature in common, the resonance form in the nonlinearity. Let us rewrite Eq. (8) in the vicinity of a resonance by using reduction $\partial^2\Phi/\partial t^2 = 0$ which denotes that the confining harmonic potential is vanishing. To suppose that confining harmonic potential is vanishing implies that the control function $\Phi(t)$ is defined as $\Phi(t) = C_0 t$ (with initial condition $\Phi(0) = 0$). Because of this, the nonlinearity in Eq. (8) has the simple dispersive form $R(t) = R_0/(1 - C_0 t)$, as also is the scattering length $a_s(t)$ (see Eq. (18)). In fact, the scattering length $a_s(t)$ can be continuously tuned by varying (linearly increasing in time) the external magnetic field $B(t) = C_1 t$ near the Feshbach resonances [19]. Thus we can write

$$i\frac{\partial Q}{\partial t} + \frac{\sigma}{2}\frac{\partial^2 Q}{\partial x^2} + \frac{R_0}{1 - C_0 t} |Q|^2 Q = 0 \tag{19}$$

Hence the nonlinear BEC behavior in the vicinity of the Feshbach resonance (18) is described by the exactly integrable NLSE model (19) where the nonlinearity has the dispersive form. Bright and dark hyperbolically growing soliton solutions for this model are given by formulas (9-14) where the substitution $\Phi(t) = C_0 t$ is trivial. This remarkable result shows that in the vicinity of the Feshbach resonance bright and dark solitons can be stabilized even without a trapping potential and, in addition, indicates the possibility for the optimal compression of bright and dark solitons in BEC. Singularity $1 - C_0 t = 0$ means that the determination of the utmost energy and the limiting pulsewidth of a soliton requires additional investigations based, e.g., on the study of three-body recombination to molecular states.

Case 3. Exactly integrable GP models with sign-reversal harmonic potentials. The case of periodically varying magnetic field $B(t) = B_1 \sin(\varpi t)$ can be regarded as a special case. In this case the spectral parameter of the IST method is directly related to the magnetic field variations $\lambda_1(t) = \Phi_t/(1 - \Phi)$ (where the control function $\Phi(t)$ is given by $\Phi(t) = \alpha \sin \varpi t$) and the parameter functions for varying in time nonlinearity and confining potential cannot be chosen independently; they satisfy the exact integrability conditions:

$$R(t) = \frac{R_0}{1 - \alpha \sin \varpi t};\ \Omega^2(t) = -\frac{\alpha\sigma\varpi^2 \sin \varpi t}{(1 - \alpha \sin \varpi t)} \tag{20}$$

Consequently, variations of confining harmonic potential are bound to be sign-reversal (periodic attractive and repulsive) to support the stable nonlinear soliton management scenario. Exact bright and dark soliton solutions for this case are represented by formulas (9-14) where the substitution $\Phi(t) = \alpha \sin \varpi t$ is straightforward.

The interested reader can take different management functions $R(t)$ or $\Phi(t)$ (see Eqs. (2, 8, 16)) to find the novel "soliton islands" in a "sea of nonlinear waves" for Bose-Einstein condensates by using the algorithm developed in this paper. The remarkable fact is that analytical solutions (9-14) are obtained here in quadratures. Their pure soliton-like features (elastic character of interaction) are confirmed by the accurate direct computer simulations. We will present the most interesting (from application point of view) examples in a separate publication. Notice that soliton solutions exist only under certain conditions and the parameter functions describing the nonlinearity $R(t)$ and confining potential $V(x,t) = 1/2\Omega^2(t)x^2$ cannot be chosen independently; they satisfy equation systems (15, 17). *Matter-wave solitons management concept must be consistent with variations of confining potential* (15, 17). Fundamental bright and dark soliton solutions to the generalized Gross-Pitaevskii nonlinear models provide a theoretical basis for the matter waves management concept and open novel opportunities for controllable creation of bright and dark solitonic matter waves and their optimal compression in Bose-Einstein condensates. The results reported in this Letter are of general physics interest and offer many opportunities for further scientific studies. For example, sign-reversal scattering lengths achieved by tuning through a Feshbach resonance by controllable way (see Eqs. (8, 19)) make it possible to investigate experimentally the self-compression of bright and dark matter waves and realize the so-called nonlinear soliton pairing [26] between them (one soliton is located on the side of repulsive interaction, another one on the side of attractive interaction relative to the Feshbach resonance). It should be emphasized that the exact integrability of the model (2) and novel soliton solutions provide novel experimental opportunities not only for the BEC physics but for the optical soliton physics as well [27].

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